Coloring Geometric Intersection Graphs

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Jagiellonian University in Kraków

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Problems and Results

Chromatic number vs clique number

Chromatic number χ: the minimum number of colors in a proper coloring of the graph

Clique number ω: the maximum size of a clique in the graph

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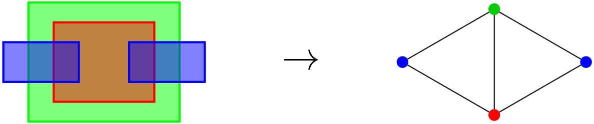
In many natural classes of graphs, $\chi \leqslant f(\omega)$ for some function $f\colon \mathbb{N} \to \mathbb{N}$.
Examples will follow. . .

Intersection and overlap graphs

Intersection graph of a family of sets $\mathcal{F}$: vertices — $\mathcal{F}$, edges — pairs of intersecting members of $\mathcal{F}$



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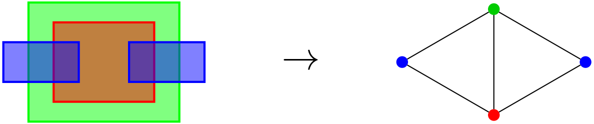
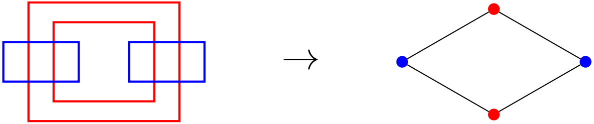
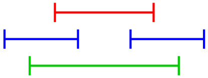


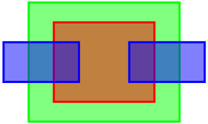
Overlap graph of a family of sets $\mathcal{F}$: vertices — $\mathcal{F}$,
edges — pairs of overlapping (intersecting but not nested) members of $\mathcal{F}$



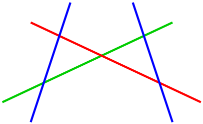
Geometric intersection and overlap graphs



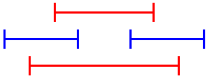
interval graphs



rectangle graphs



segment graphs



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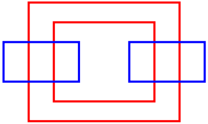

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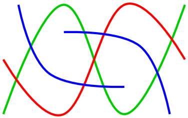

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G is perfect if and only if its complement $\overline{G}$ is perfect. (Lovász, 1972)

G is perfect if and only if neither of G, $\overline{G}$ contains an induced cycle of odd length $\geqslant 5$. (Chudnovsky, Robertson, Seymour, Thomas, 2006)

Near-perfect classes of graphs

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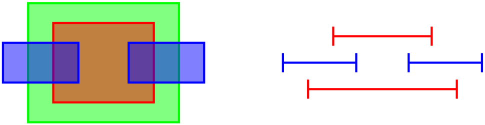

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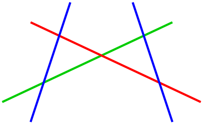

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Intersection graphs of curves each crossing a fixed curve in $1 \leqslant ? \leqslant t$ points are near-perfect, for every t. (Rok, W, 2016+)

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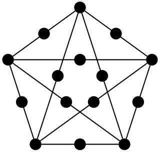

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$\to$ Yes if excluding odd length $\geqslant 5$ (Scott, Seymour, 2016+)

$\to$ Yes if excluding length $\geqslant k$ (Chudnovsky, Scott, Seymour, 2016+)

Bounds on chromatic number

	construction	upper bound
rectangle graphs	$\Theta(\omega)$	$O(\omega^2)$ (Asplund, Grünbaum, 1960)
interval overlap graphs	$\Theta(\omega \log \omega)$ (Kostochka, 1988)	$O(\omega^2 4^\omega)$ (Gyárfás, 1985) $O(\omega^2 2^\omega)$ (Kostochka, 1988) $O(2^\omega)$ (Kostochka, Kratochvíl, 1997)
discs, squares, fat objects	$\Theta(\omega)$	average degree $\Theta(\omega)$ (Pach, 1980)

Asymptotic growth of chromatic number

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There are segment graphs with $\omega = 2$ and $\chi = \Theta(\log \log n)$.
There are rectangle overlap graphs with $\omega = 2$ and $\chi = \Theta(\log \log n)$.
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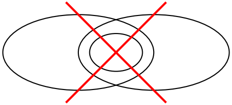
A clean overlap graph has an overlap model
with no set contained in two overlapping sets.



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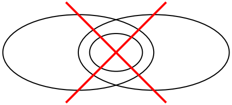

Rectangle overlap graphs with $\omega = 2$ satisfy $\chi = O(\log\log n)$.
(Krawczyk, Pawlik, W, 2013)

Clean rectangle overlap graphs satisfy $\chi = O_\omega(\log\log n)$.
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Asymptotic growth of chromatic number

	construction	upper bound
clean rectangle overlap graphs	$\Theta(\log\log n)$ for $\omega = 2$ (Pawlik et al., 2013–14)	$O_\omega(\log\log n)$ (Krawczyk, W, 2014)
rectangle overlap graphs		$O_\omega((\log\log n)^{\omega-1})$ (Krawczyk, W, 2014)
segment graphs		$O_\omega(\log n)$ (Suk, 2014)
string graphs	$\Theta_\omega((\log\log n)^{\omega-1})$ (Krawczyk, W, 2014)	$(\log n)^{O(\log\omega)}$ (Fox, Pach, 2014)

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How about rectangle overlap/segment/1-intersecting string graphs?

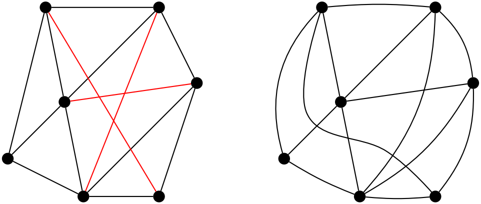
Do they satisfy $\chi_3 \leqslant f(\omega)$ for some $f\colon \mathbb{N} \to \mathbb{N}$?

If yes, this would (partially) solve the next problem...

Quasi-planarity

Topological/geometric graph: graph drawn in the plane with edges represented by curves/straight-line segments.

Such a graph is *k*-quasi-planar if it has no *k* pairwise crossing edges.



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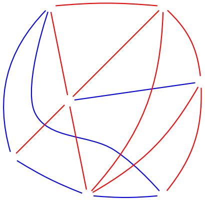
General bounds on the number of edges in k-quasi-planar graphs:

- $O_k(n \log n)$ for geometric graphs (Valtr, 1997)
- $O_k(n \log n)$ for 1-intersecting topological graphs (Suk, W, 2013)
- $O_{k,t}(n \log n)$ for t-intersecting topological graphs (Rok, W, 2016+)
- $n(\log n)^{O(\log k)}$ for topological graphs (Fox, Pach, 2012)

Methods

Quasi-planarity via coloring

The edges are properly/K_3-free/K_4-free colorable using $f(k, n)$ colors;



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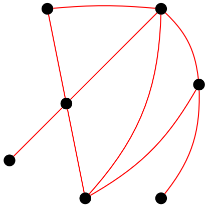

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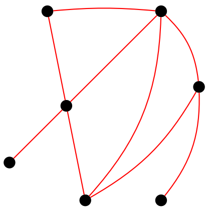

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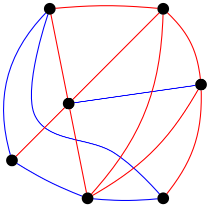

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Segment intersection graphs satisfy $\chi_3 \leqslant f(\omega)$; (conjecture)
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$\rightarrow$ k-quasi-planar geometric graphs have $O_k(n \log n)$ edges.

Segment intersection graphs satisfy $\chi_3 \leqslant f(\omega)$; (conjecture)
$\rightarrow$ k-quasi-planar geometric graphs have $O_k(n)$ edges.

Intersection graphs of curves each crossing a fixed curve in
$1 \leqslant ? \leqslant t$ points satisfy $\chi \leqslant f(\omega, t)$; (Rok, W, 2016+)
$\rightarrow$ k-quasi-planar t-intersecting topological graphs have $O_{k,t}(n)$ edges crossing any fixed edge;
$\rightarrow$ (Fox, Pach, Suk, 2013) they have $O_{k,t}(n \log n)$ edges in total.

On-line graph coloring

Presenter builds a graph presenting vertices one by one with all edges connecting them to previous vertices.
Algorithm colors each vertex immediately after it is presented.

Algorithm wants to use as few colors as possible.
Presenter wants to force Algorithm to use as many colors as possible.

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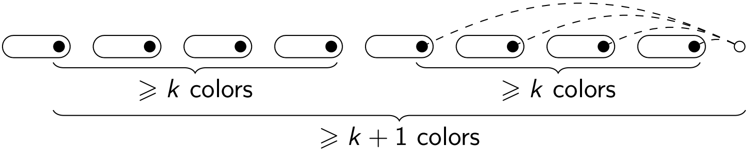
Presenter can force $\lfloor \log_2 n \rfloor + 1$ colors on a forest: (Bean, 1976)



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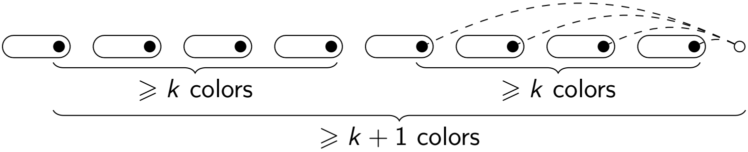

First-fit uses $\leqslant \lfloor \log_2 n \rfloor + 1$ colors on any forest. (folklore)

On-line coloring of clean interval overlap graphs

Presenter builds a clean interval overlap graph presenting intervals one by one in the increasing order of left endpoints.
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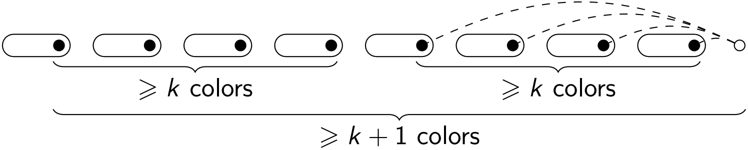
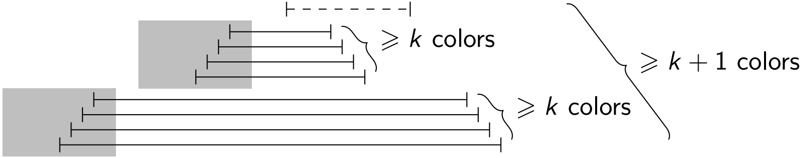
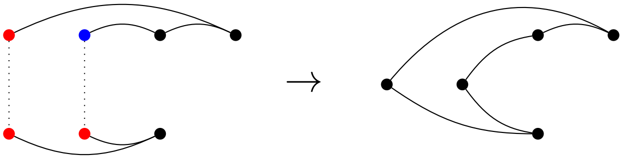




On-line vs off-line coloring

Encoding Presenter's strategy in a single game graph:



$\omega \leqslant k$
forcing m colors

$\leftrightarrow$

$\omega \leqslant k$
$\chi \geqslant m$

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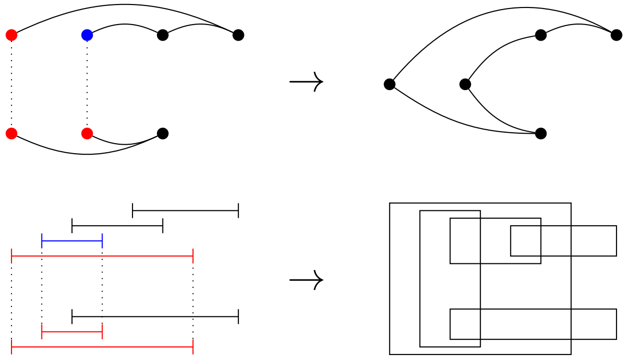

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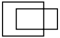
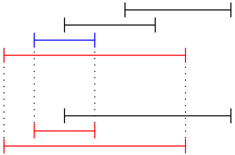
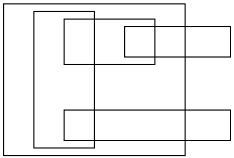
Presenter builds a clean interval overlap graph presenting intervals one by one in the increasing order of left endpoints.

Presenter can force $\lfloor \log_2 n \rfloor + 1$ colors on a forest; (Pawlik et al., 2013–14)

$\rightarrow$ (encoding) there are clean rectangle overlap graphs with all overlapping pairs of the form , with $\omega = 2$ and $\chi = \Theta(\log \log n)$;



$\rightarrow$



$\omega \leqslant k$ forcing m colors	$\leftrightarrow$	$\omega \leqslant k$ $\chi \geqslant m$

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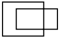
$\rightarrow$ (encoding) there are clean rectangle overlap graphs with all overlapping pairs of the form , with $\omega = 2$ and $\chi = \Theta(\log \log n)$;

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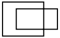
$\rightarrow$ (encoding) there are clean rectangle overlap graphs with all overlapping pairs of the form , with $\omega = 2$ and $\chi = \Theta(\log \log n)$;

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Algorithm can do with $\leqslant \lfloor \log_2 n \rfloor + 1$ colors on any forest; (folklore)

$\rightarrow$ (some reductions) Algorithm can do with $O_\omega(\log n)$ colors;

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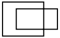
$\rightarrow$ (encoding) there are clean rectangle overlap graphs with all overlapping pairs of the form , with $\omega = 2$ and $\chi = \Theta(\log\log n)$;

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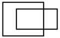
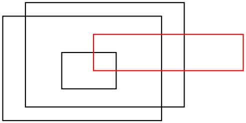
$\rightarrow$ (game interpretation) (heavy-light decomposition, Sleator, Tarjan, 1976) clean rectangle overlap graphs with all overlapping pairs of the form  satisfy $\chi = O_\omega(\log\log n)$.

Breadth-first search

Triangle-free overlap graphs

Run BFS from an outermost vertex;

$\rightarrow$ every level is clean.



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Run BFS from an outermost vertex;
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Color every level separately, repeating colors every 2 levels;
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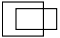
Overlap graphs with $\omega = k$

Run k-clique BFS from an outermost vertex;
$\to$ every level is clean.

Color every level separately, repeating colors every 2 levels;
$\to$ every color class is a graph with $\omega \leqslant k - 1$.

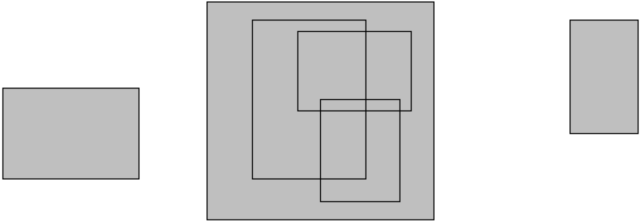
Combining various coloring results

Clean rectangle overlap graphs with all overlapping pairs of the form  satisfy $\chi = O_\omega(\log\log n)$. (shown before)

Rectangle intersection graphs are near-perfect. (Asplund, Grünbaum, 1960)

Outerstring graphs are near-perfect. (Suk, 2014; Rok, W, 2014)

$\rightarrow$ Clean rectangle overlap graphs satisfy $\chi = O_\omega(\log\log n)$.



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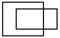
Clean rectangle overlap graphs with all overlapping pairs of the form  satisfy $\chi = O_\omega(\log \log n)$. (shown before)

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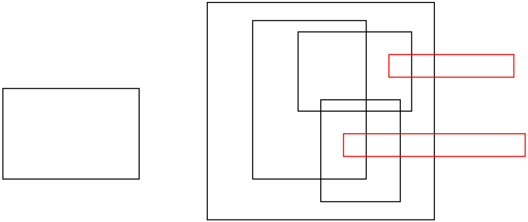

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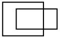
Clean rectangle overlap graphs with all overlapping pairs of the form  satisfy $\chi = O_\omega(\log\log n)$. (shown before)

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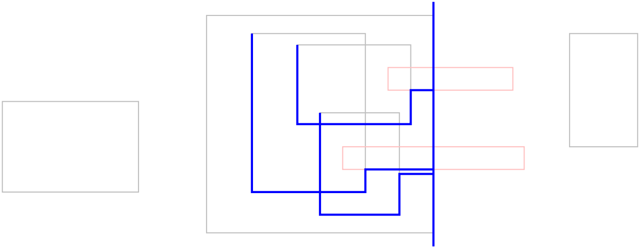

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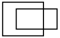
Clean rectangle overlap graphs with all overlapping pairs of the form  satisfy $\chi = O_\omega(\log\log n)$. (shown before)

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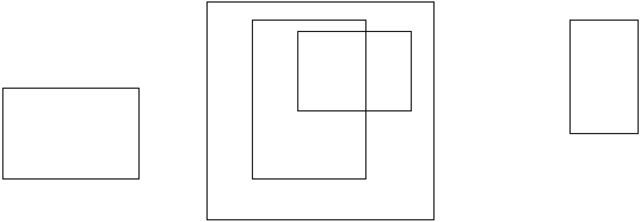

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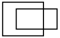
Clean rectangle overlap graphs with all overlapping pairs of the form  satisfy $\chi = O_\omega(\log\log n)$. (shown before)

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Coloring of clean subgraphs $\rightarrow$ reduction to graphs with smaller ω.

$\rightarrow$ (induction) Rectangle overlap graphs satisfy $\chi = O_\omega((\log\log n)^{\omega-1})$.

Radius 2 subgraphs

Every string graph with large χ has a subgraph with radius 2 and large χ.
(Chudnovsky, Scott, Seymour, 2016+)

$\rightarrow$ (very long and convoluted argument)
Intersection graphs of curves each crossing a fixed curve in $1 \leqslant ? \leqslant t$ points are near-perfect, for every t. (Rok, W, 2016+)

Problems

1. Improve the bounds:

	construction	upper bound
rectange graphs	$\Theta(\omega)$	$O(\omega^2)$
interval overlap graphs	$\Theta(\omega \log \omega)$	$O(2^\omega)$
rectangle overlap graphs	$\Theta(\log \log n)$ for $\omega = 2$	$O_\omega((\log \log n)^{\omega-1})$
segment graphs		$O_\omega(\log n)$
string graphs	$\Theta_\omega((\log \log n)^{\omega-1})$	$(\log n)^{O(\log \omega)}$

2. Do rectangle overlap/segment graphs satisfy $\chi_3 \leqslant f(\omega)$?

3. Do k-quasi-planar geometric/topological graphs have $O_k(n)$ edges?